Some dynamical properties of the stadium billiard

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Abstract

We consider the dynamical properties of one kind of cyclically ordered orbits of the stadium billiard. We prove that the stadium billiard possesses infinitely many such orbits, that they are hyperbolic, and that their stable and unstable manifolds intersect transversely. Hence the stadium billiard contains a Smale horseshoe. We construct symbolic dynamics for these orbits and consider their sensitivity to choice of initial condition by studying their rotation numbers.

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1. Introduction

The classical billiard system describes the free motion of a particle in a planar region bounded by a closed curve. The particle moves along a straight line and is reflected from the boundary according to the rule “the angle of reflection equals the angle of incidence”. The systematic study of the classical billiard was started by Birkhoff to illustrate and develop some concepts in the theory of Hamiltonian dynamical systems with two degrees of freedom [3]. Since then, the billiard has become a basic model in such diverse fields as foundations of statistical mechanics, ergodic theory, quantum chaos, etc. Billiards represent the simplest systems in classical mechanics that exhibit the full range of behavior observed in two-degree-of-freedom Hamiltonian systems. Thus the study of billiards problem can be helpful for understanding complex behavior of Hamiltonian systems.

The billiard defines a two-dimensional discrete dynamical system which may present different dynamical behavior according to the boundary. For example, if the boundary is a circle or an ellipse, the corresponding dynamical system is completely integrable, and the dynamical behavior is regular. However, in most cases, the billiard mapping is no longer integrable, and may possess complex dynamical behavior, for example, chaotic, ergodic, etc.
In this paper we are interested in the stadium billiard whose boundary $\Gamma$ consists of two half circles of radius 1 slid apart a distance $2h$, and joined by two straight segments of equal length (see Fig. 1). The stadium billiard had been studied by many authors [2,4,5,9]. Bunimovich [4,5] proved that the stadium billiard is a completely chaotic system, that almost all orbits have positive Lyapunov exponents. However, due to the difficulty for expressing the map in explicit form, other than general results, there are few results on a detailed description of the dynamics of the stadium billiard. In some special cases, one can construct symbol dynamical system and Markov partitions for the stadium billiard [2,11]. In this paper, we will consider one kind of special orbits of the stadium billiard, and give some statements of the chaos of these orbits.

Using Birkhoff’s coordinate $(s, \theta)$, where $s \in [0, L)$ is the arc length along the boundary $\Gamma$ measured from one fixed point, $A$ say, and $\theta$ is the angle between the reflection direction and the tangent vector at the impact point, the billiard moving in the stadium defines a map $\Psi$ from the annulus $\mathcal{A} = [0, L) \times (-\pi/2, \pi/2)$ into itself. Let $(s_0, \theta_0) \in \mathcal{A}$ and $(s_1, \theta_1) \in \mathcal{A}$ be such that $\Psi(s_0, \theta_0) = (s_1, \theta_1)$ and that $\Gamma$ is $C^\infty$-diffeomorphism in some neighborhoods of $s_0$ and $s_1$, respectively (notice that $\Gamma$ is globally $C^1$ but not $C^2$ and is piecewise $C^\infty$). Then $\Psi$ is $C^\infty$-diffeomorphism in some neighborhoods of $(s_0, \theta_0)$ and $(s_1, \theta_1)$. It also preserves the measure $d\mu = \cos \theta \, d\theta \, ds$. Then $(\mathcal{A}, \mu, \Psi)$ defines a discrete dynamical system, whose orbits are given by

$$O = \{(s_k, \theta_k) = \Psi^k(s_0, \theta_0), \quad k \in \mathbb{Z}\} \subset \mathcal{A}. \quad (1)$$

Hence the stadium billiard defines a one-parameter family of diffeomorphisms $\Psi_h$ whose dynamics depend on the values of $h$. For instance, when $h = 0$, the stadium become the unit circle, and the map $\Psi_0$ is given explicitly by

$$s_1 = s_0 + 2h \theta_0, \quad \theta_1 = \theta_0.$$ 

This system is completely integrable, and every orbit stays in an invariant closed curve $\theta = \text{const.}$ and is cyclically ordered. When $h > 0$, Bunimovitch’s results show that almost all orbits are ‘chaotic’ [4]. However, the map $\Psi_h$, which is the perturbation of the completely integrable system $\Psi_0$, has many quasi-periodic orbits for any $h > 0$ by Aubry–Mather theory [8] as well.

The orbits of $\Psi_h$ that are most like those of the un-perturbed system $\Psi_0$ are cyclically ordered orbits. In this paper we will consider one special kind of cyclically ordered orbits. It is evident that, for a cyclically ordered orbit, there are at most one sequential point lying on the straight segment. The orbits we considered are cyclically ordered orbits with exactly one point on a straight segment between each pair of points in different half circles, said 1-CO orbit in short (see Fig. 1).

1 The authors thank the referees for their suggestion on the definition of 1-CO orbit.
This paper is organized as follows. In Section 2 we will prove that the stadium billiard possesses infinitely many hyperbolic periodic 1-CO orbits. Moreover, for any $h > 0$, there exist stable and unstable manifolds of these periodic orbits that intersect transversely (Theorem 3). Consequently, there is chaos in the stadium billiard with horseshoe map’s type, i.e., exist dense periodic orbits and non-periodic orbits, sensitive dependence of initial values. In Section 3, we will introduce a symbolic dynamical system for 1-CO orbits, and prove the following fact (Theorem 13):

For any two side sequence of positive integers $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \omega_2, \ldots)$ with some geometrical restrictions on $\omega_i$, there exists a unique 1-CO orbit, begun from a point at a straight segment, followed by $\omega_0$ points at one half circle segment, then one point at other straight segment, then $\omega_1$ points at other half circle segment, then one point at original straight segment, and so on, and the same for the inverse direction. Such kind of chaos is familiar in many problems. For example, in the forced pendulum [6,10]. We also consider rotation numbers of the 1-CO orbits in this section. We will prove that rotation numbers of 1-CO orbits depend on their initial values sensitively.

2. Invariant set of the map $\tilde{\Psi}$

Due to symmetry of the stadium, we can identity $I_1 = \overrightarrow{AB}$ with $I_3 = \overrightarrow{A'B'}$. Hereinafter we will denote $I_1$ or $I_3$ by $I = [0, 2h]$ without ambiguity. While considering 1-CO orbits, we will study, instead of the map $\Psi$ itself, the map $\tilde{\Psi}$:

\[ \tilde{\Psi} : I \times \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \mapsto I \times \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), \]

which is induced from the map $\Psi$. For example, let $(s_0, \theta_0) \in I_1 \times (-\pi/2, \pi/2)$, then $\tilde{\Psi}$ maps it to the successive point $(s_k, \theta_k) \in I_3 \times (-\pi/2, \pi/2)$ of the orbits started from the point $(s_0, \theta_0)$.

Now we give an explicit expression of $\tilde{\Psi}$. It is necessary for the orbit started from $(s, \theta) \in I \times (-\pi/2, \pi/2)$ to be 1-CO that the angle $\theta$ satisfies

\[ 0 < \theta < \min\{\arctan(2h-s)^{-1}, \arctan s^{-1}\}. \]

Let

\[ \mathcal{A}_0 = \{(s, \theta) | 0 \leq s \leq 2h, 0 < \theta < \min\{\arctan(2h-s)^{-1}, \arctan s^{-1}\}\}, \]

then $\tilde{\Psi}$ maps $\mathcal{A}_0$ into $[0, 2h] \times [-\pi/2, \pi/2]$. Define

\[ \psi(s, \theta) = \arccos(\cos \theta - s \sin \theta), \]

\[ \Theta(s, \theta) = 2\psi(2h-s, \theta), \]

\[ X(s, \theta) = \frac{\cos \Theta(s, \theta) - \cos \psi(s, \theta)}{\sin \Theta(s, \theta)}. \]

(2)

where $\lfloor x \rfloor$ denotes the largest integer not more than $x$ and $\{x\} = x - \lfloor x \rfloor$. Then $\tilde{\Psi}$ is given by

\[ \tilde{\Psi}(s, \theta) = (X(2h-s, \theta), \Theta(2h-s, \theta)). \]

(3)

By the symmetry, the inverse of $\tilde{\Psi}$ is given by

\[ \tilde{\Psi}^{-1}(s, \theta) = (2h - X(s, \theta), \Theta(s, \theta)). \]

(4)
Obviously
\[ N(s, \theta) = \left[ \frac{\pi - \theta}{2(h - s, \theta)} \right] \tag{5} \]
is exactly the number of points between the sequential impacted positions \((s, \theta)\) and \(\tilde{\Psi}(s, \theta)\) within the orbit of the stadium billiard.

Let (1) be the 1-CO orbit started from \((s_0, \theta_0)\), there is a corresponding orbit defined by \(\tilde{\Psi}\) (here \((s, \theta)\) := \((s_0, \theta_0)\)):
\[ \tilde{O} = \{(s_i, \theta_i) = \tilde{\Psi}(s_i, \theta_i), \quad i \in \mathbb{Z}\} \subset A_0. \tag{6} \]
Then \(\tilde{O}\) is the intersection of \(O\) with \(A_0\). Hence, some dynamical properties of \(O\) can be studied through that of \(\tilde{O}\).

### 2.1. Fixed points

It is not difficult to see that all fixed points of \(\tilde{\Psi}\) are given by \(p_n = (h, \theta_n)\), where \(\theta_n\) satisfies the equation
\[ \theta + n \arccos \left( \frac{\cos \theta - h \sin \theta}{\sin \psi} \right) = \frac{\pi}{2} \quad (n \in \mathbb{N}). \tag{7} \]

For any \(n \in \mathbb{N}\), there exists a unique \(\theta_n \in (0, \arctan h - 1)\) satisfying (7) with \(n = N(p_n)\). Moreover, the sequence \(\{\theta_n\}\) tends to zero monotonically. Now we show that these fixed points are hyperbolic with reflection. Let \((s', \theta') = \tilde{\Psi}(s, \theta), \quad \psi = \varphi(2h - s, \theta), \quad n = N(2h - s, \theta), \quad a(s, \theta) = 1 + s \cot \theta.\]

Then the Jacobian of \(\tilde{\Psi}\) is given by \(J(s, \theta) = (J_{ij}(s, \theta))_{2 \times 2}\) with
\[ J_{11}(s, \theta) = -2a(s', \theta') \frac{\sin \theta}{\sin \psi}, \quad J_{12}(s, \theta) = a(s', \theta') \left( 1 + 2n \arcsin (2h - s, \theta) \frac{\sin \theta}{\sin \psi} \right) + a(2h - s, \theta) \frac{\sin \theta}{\sin \psi}, \]
\[ J_{21}(s, \theta) = 2a(s', \theta') \frac{\sin \theta}{\sin \psi}, \quad J_{22}(s, \theta) = 1 - 2n \arcsin (2h - s, \theta) \frac{\sin \theta}{\sin \psi}. \tag{8} \]

We have
\[ \text{Det } J(s, \theta) = \frac{\sin \theta}{\sin \psi}, \quad \text{Tr } J(s, \theta) = -\left( 1 + \frac{\sin \theta}{\sin \psi} + 2n \frac{\sin \theta}{\sin \psi} (a(s', \theta') + a(2h - s, \theta)) \right), \tag{9} \]
\[ \text{Det } J(p_n) = 1, \quad \text{Tr } J(p_n) = -2 \left( 1 + 2n \arcsin (h, \theta_n) \frac{\sin \theta_n}{\sin \psi(h, \theta_n)} \right) < -2. \tag{10} \]
Thus the eigenvalues of \(J(p_n)\) satisfy \(\lambda_1 < -1 < \lambda_2 < 0\), which implies that the fixed points are hyperbolic with reflection (inverted saddle).

**Theorem 1.** The map \(\tilde{\Psi}\) has countable fixed points \(p_n = (h, \theta_n)\), with the sequence \(\{\theta_n\}\) tending to zero monotonically, and all fixed points are hyperbolic with reflection.

**Remark 2.** The fixed point \((h, \theta_0)\) of \(\tilde{\Psi}\) corresponds to a \((2n + 2)\)-periodic point of the stadium billiard map \(\Psi\). In some neighborhood of \((h, \theta_0)\), we have \(\psi^{2n+2} = \tilde{\Psi}^2\). Hence, the periodic point is exactly hyperbolic fixed point of \(\Psi^{2n+2}\), i.e., the periodic orbit is hyperbolic.
2.2. Stable manifolds and unstable manifolds of the fixed points

In this subsection we consider the stable manifolds and unstable manifolds of the fixed points. The main result is the following theorem.

**Theorem 3.** There exists a sequence \( \{h_n\} \) which tends to zero monotonically, such that if \( h \geq h_n \), then for any \( m \geq n \), the stable (unstable) manifold of \( p_m \) and the unstable (stable) manifold of \( p_{m+1} \) intersect transversely.

**Remark 4.** From the discussion below, \( h_n \) can be given by

\[
h_n = \frac{1}{2} \cot \left( \frac{\pi}{2n+4} \right) - \frac{1}{2} \left[ \cos \left( \frac{\pi}{2n+4} \right) \csc \left( \frac{\pi}{2n+4} \right) \right],
\]

(11)

and \( h_1 = \frac{\sqrt{3} - 1}{2} \).

We have the following Corollaries immediately [10, Section 6.5].

**Corollary 5.** If \( h \geq h_1 = \frac{\sqrt{3} - 1}{2} \), then for any \( n \in \mathbb{N} \), the stable (unstable) manifold of \( p_n \) and unstable (stable) manifold of \( p_{n+1} \) intersect transversely.

**Corollary 6.** For any \( h > 0 \), there exists a Cantor set contained in the invariant set of \( \tilde{\Psi} \), and some \( m \in \mathbb{N} \), such that \( \tilde{\Psi}^m \) restricted to the Cantor set, is isomorphic to a horseshoe map.

Before proving Theorem 3, we define some functions as follows. Let

\[
F_{h,n}(s, \theta) = 2h \sin (\theta + 2n \varphi(s, \theta)) + \cos (\theta + 2n \varphi(s, \theta)) + \cos \varphi(s, \theta),
\]

\[
G_{h,n}(s, \theta) = \theta + (2n + 1) \varphi(s, \theta) - \pi,
\]

\[
K_{h,n}(s, \theta) = \theta + 2n \varphi(s, \theta) - \pi
\]

for any \( n \in \mathbb{N} \) and \( h > 0 \), and \( f_{h,n}(s), g_{h,n}(s), k_{h,n}(s) \in (0, \arctan s^{-1}) \) be the functions satisfying the following equations:

\[
F_{h,n}(s, f_{h,n}(s)) \equiv 0, \quad G_{h,n}(s, g_{h,n}(s)) \equiv 0, \quad K_{h,n}(s, k_{h,n}(s)) \equiv 0, \quad \forall s \in [0, 2h].
\]

(12)

Then, we have the following lemma.

**Lemma 7.** There exist unique function sequences \( \{k_{h,n}(s)\}_{n \in \mathbb{N}}, \{f_{h,n}(s)\}_{n \in \mathbb{N}}, \{g_{h,n}(s)\}_{n \in \mathbb{N}} \) for \( s \in [0, 2h] \), satisfying (12) and \( f_{h,n}(s), g_{h,n}(s), k_{h,n}(s) \in (0, \arctan s^{-1}) \). Moreover, the functions are monotonically decreasing with respect to \( s \), and have the following order relation:

\[
\arctan s^{-1} > k_{h,1}(s) > f_{h,1}(s) > g_{h,1}(s) > \cdots > k_{h,n}(s) > f_{h,n}(s) > g_{h,n}(s) > \cdots > 0.
\]

(13)

The proof of Lemma 7 is elementary by Implicit Function Theorem and Intermediate Value Theorem.

From Lemma 7, let

\[
U_s = \{(s, \theta) | 0 \leq s \leq 2h, \quad g_{h,n}(2h - s) \leq \theta \leq f_{h,n}(2h - s)\},
\]

\[
V_s = \{(s, \theta) | 0 \leq s \leq 2h, \quad g_{h,n}(s) \leq \theta \leq f_{h,n}(s)\}
\]

be strips contained in \( A_n \), then \( \tilde{\Psi}(U_s) = V_s \), \( \tilde{\Psi}(\partial U_s) = \partial V_s \). Moreover, \( \tilde{\Psi} \) maps \( A_s, B_s, C_s, D_s \) to \( A'_s, B'_s, C'_s, D'_s \), respectively (see Fig. 2).
Before some further discussions, we introduce some definitions.

**Definition 8.** We say that the graph of a continuous function \( \theta = u(s) \) is an up (or down) horizontal curve in \( A_0 \) if \( (s, u(s)) \in A_0 \) for any \( s \in [0, 2h] \), and \( u(s) \) is monotonically increasing (or decreasing). If the two up (or down) horizontal curves \( \theta = u_1(s) \) and \( \theta = u_2(s) \) satisfy
\[
(s, u_i(s)) \in A_0, \quad u_1(s) < u_2(s), \quad 0 \leq s \leq 2h,
\]
we say that the set
\[
U = \{(s, \theta)| 0 \leq s \leq 2h, u_1(s) \leq \theta \leq u_2(s)\}
\]
(14)
is an up (or down) horizontal strip. The diameter of an up (or down) horizontal strip is defined as
\[
d(U) = \max_{0 \leq s \leq 2h} (u_2(s) - u_1(s)).
\]

Let \( U \) be the up horizontal strip defined by (14), and \( V \) be the down horizontal strip defined by
\[
V = \{(s, \theta)| 0 \leq s \leq 2h, v_1(s) \leq \theta \leq v_2(s)\}.
\]
(15)
If \( v_1(0) \geq u_2(0), v_2(2h) \leq u_1(2h) \), we say that \( U \) is transverse with \( V \). Similarly, we say that a curve \( \theta = u(s) \) is transverse with \( V \), if \( v_1(0) \geq u(0), v_2(2h) \leq u(2h) \). 

**Lemma 9.** For any \( h > 0 \), \( U_h \) is transverse with \( V_0 \).

**Proof.** We only need to prove \( f_{k+1}(2h) < g_{k+1}(0) \). Set \( f_k(h) = f_{k+1}(2h) \). From the uniqueness of \( f_{k+1}(s) \), the function \( f_k(h) \) satisfies
\[
\pi - (\theta + 2\phi(2h, \theta)) = \theta.
\]
Since \( \phi(2h, f_k(h)) > f_k(h) \), we have \( \pi > 2(\pi + 1)f_k(h) \), and hence \( f_k(h) < \pi/(2\pi + 2) = g_{k+1}(0) \).

**Lemma 10.** Let \( h_n \) be defined as (11), then
(1) the sequence \([h_n] \) tends to zero monotonically;
(2) for any \( h \geq h_n \), \( U_h \) (or \( V_h \)) is transverse with \( V_{h+1} \) (or \( U_{h+1} \)).
Proof.

(1) From the expression of $h(n)$, it follows that

$$h'(n) = \frac{\pi}{4n(n+2)} \left( n^2 + (n+1) \cos \frac{(2n+1)\pi}{2n} - (n^2 + n+1) \cos \frac{\pi}{2n(n+2)} \right)$$

Note that for $n \geq 1$ the inequality

$$\frac{\pi}{2} > \frac{(n+1)\pi}{2n(n+2)}$$

holds. Therefore

$$n^2 + (n+1) \cos \frac{(2n+1)\pi}{2n(n+2)} - (n^2 + n+1) \cos \frac{\pi}{2n(n+2)}$$

Thus, $h(n) < 0$, which implies that $\{h_n\}$ is monotonically decreasing. The proof of $\{h_n\}$ tending to 0 is elementary.

(2) We need to show that, for $h \geq h_n$, $f_n(h) = f_{g_{n+1}}(2h) \leq g_{n+1}(0)$. The function $f_n(h)$ satisfies

$$2f_n(h) + 2\psi(2h, f_n(h)) = \pi,$$

which implies that

$$f_n(0) = \frac{\pi}{2n+2}, \quad f'_n(h) < 0, \quad \lim_{h \to +\infty} f_n(h) = 0.$$

Thus there exists a unique $h_n$ such that

$$f_n(h_n) = \frac{\pi}{2n+4} = g_{n+1}(0).$$

Substituting it into (16), and solving for $h_n$, we obtain

$$h_n = \frac{1}{2} \cot \frac{\pi}{2n+4} - \frac{1}{2} \cot \frac{(n+1)\pi}{2n(n+2)} \csc \frac{\pi}{2n+4}$$

Obviously, for any $h \geq h_n$, $f_n(h) \leq f_n(h_n) = g_{n+1}(0).$

Lemma 11. Let $V_0^n = U_n$, $V_{n+1}^n = \tilde{\Psi}(V_n^{n-1} \cap U_n)$. For any $m \geq 1$, we have

1. $V_{n+1}^m \subset V_n^m$, and $V_n^m$ is a down horizontal strip which is transverse with $U_n$;
2. $d(V_{n+1}^m) \leq \frac{1}{(2n+1)d(V_{n+1}^m)}$;
3. $W_n^m := \cap_{m \geq 1} V_n^m$ is a down horizontal curve contained in $V_n$. 

\[\square\]
Proof.

(1) Firstly, \( V_1 \) and \( V_2 \) are up and down horizontal strips, respectively. From Lemma 9, \( V_1 \) is transverse with \( U_0 \). By induction, we assume that \( V_{1n-1} \subset V_{1n} \subset V_1 \), then \( V_{1n} = \Psi(V_{2n-1} \cap U_0) \subset \Psi(V_{2n-2} \cap U_0) = V_{2n} \subset V_2 \) and \( V_n \) is transverse with \( U_0 \).

(2) We will prove that for any down horizontal strip \( V \) which is transverse with \( U_0 \), have \( d\Psi(V \cap U_0) \leq (1/2n + 1) d(V) \). Let \( V^* = \Psi(V \cap U_0) \), then \( V^* \) is a down horizontal strip. Assume that \( V \) and \( V^* \) be defined by

\[
V = \{ (s, \theta) | 0 \leq s \leq 2h, \quad v_1(s) \leq \theta \leq v_2(s) \}, \quad V^* = \{ (s, \theta) | 0 \leq s \leq 2h, \quad v_1(s) \leq \theta \leq v_2^*(s) \},
\]

respectively. Let \( z^*_1 = (s^*, v_1^*(s^*)), \quad z^*_2 = (s^*, v_2^*(s^*)) \) be two points in \( V^* \), and set the line segment as follows:

\[
z^*(t) := (1 - t)z^*_1 + tz^*_2 \quad (t \in [0, 1]).
\]

Let \( z(t) := (s(t), \theta(t)) = \tilde{\Psi}^{-1}(z^*(t)) \), then

\[
\theta(0) = v_1(s(0)), \quad \theta(1) = v_2(s(1)).
\]

We have

\[
(d\tilde{\Psi})_{z(t)}(z(t)) = z^*(t) - z^*_1 = (0, \Delta z^*),
\]

where \( \Delta z^* = v_2^*(s^*) - v_1^*(s^*) \), hence

\[
J_2(z(t))\dot{s}(t) + J_2(z(t))\dot{\theta}(t) = 0, \quad J_{12}(z(t))\dot{s}(t) + J_{12}(z(t))\dot{\theta}(t) = \Delta z^*,
\]

where \( J(z(t)) = (J_2(z(t)))_{2 \times 2} \) is the Jacobian of \( \tilde{\Psi} \) given by (8). Thus we have

\[
\Delta z^* = \left( \frac{\text{Det}(J_2(z(t)))}{J_{12}(z(t))} \right) \dot{\theta}(t).
\]

For any \( (s, \theta) \in U_0 \), we have

\[
\left| \frac{J_{12}(s, \theta)}{\text{Det}(J(s, \theta))} \right| = \left| -\frac{2na(s', \theta')}{\sin \psi(2h, \theta)} \frac{\sin \theta}{\sin \theta'} \right| = \left| \frac{\sin \theta}{\sin \theta'} \right| \left| \frac{\sin \theta}{\sin \theta'} \right| = 1, \quad (s', \theta') \in A_0.
\]

Therefore

\[
|\Delta z^*| \leq \frac{1}{2n + 1} |\dot{\theta}(t)|.
\]

From (17), \( \dot{\theta}(t) \) has fixed sign for \( t \in [0, 1] \), we obtain

\[
|v_2^*(s^*) - v_1^*(s^*)| = \int_0^1 |\Delta z^*| \, dt \leq \frac{1}{2n + 1} \int_0^1 |\dot{\theta}(t)| \, dt = \frac{1}{2n + 1} |\theta(1) - \theta(0)|.
\]
Assume that \( s(1) \geq s(0) \), then

\[
|\theta(1) - \theta(0)| = v_2(s(1)) - v_1(s(0)) \leq v_2(s(1)) - v_1(s(1)) \leq d(V),
\]

which implies \( d(V^*) \leq (1/(2n + 1))d(V) \). This completes the proof of (2).

From (1) and (2) and Theorem of Nested Intervals, (3) can be proved easily and we omit it here.

Similarly, we have the following lemma.

**Lemma 12.** Let \( U_0 = V_0, U_n = \tilde{\Psi}^{-1}(U_{n-1} \cap V_n) \). For any \( m \geq 1 \), we have

1. \( U_{n+1} \subset U_n \), and \( U_n \) is an up horizontal strip which is transverse with \( V_n \);
2. \( d(U_{n+1}) \leq (1/(2n + 1))d(U_n) \);
3. \( W^u_n := \cap_{m \geq 1} U_n \) is an up horizontal curve contained in \( U_n \).

**Proof of Theorem 3.** Note that \( W^u_n \cap W^s_n = \{p_k\} \), where \( W^u_n, W^s_n \) are subsets of the unstable and stable manifolds of \( p_n \), respectively. Since \( W^u_n \) and \( W^s_n \) are the down and up horizontal curves, respectively, they are intersect transversely if \( W^u_n \cap W^s_m \neq \emptyset \). From Lemma 10, for \( h \geq b_n \geq h_n (\forall m \geq n) \), \( V_{m+1}(U_{m+1}) \) is transverse with \( U_m \cap V_m \), which implies, from Lemmas 11 and 12, that

\[
W^u_{m+1} \cap W^s_n \neq \emptyset, \quad W^s_{m+1} \cap W^u_n \neq \emptyset.
\]

This completes the proof of Theorem 3.

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3. **Dynamical properties of 1-CO orbits**

3.1. **Symbol dynamical system for the invariant set**

Let \( A = \cap \infty \tilde{\Psi}(A_0) \) be the invariant set of \( \tilde{\Psi} \). Then \( A \) consists of the points in \( A_0 \) that contained in a 1-CO orbit. A rough statement of dynamical properties of 1-CO orbits can be presented by symbol dynamical system associated with \( \tilde{\Psi}|_A \) as follows.

**Theorem 13.** Let \( \hat{\mathcal{O}} \) be a 1-CO orbit, with \( \hat{\mathcal{O}} = \{ (x_0, \theta_0) \}_{i=-\infty}^{\infty} \) be the intersection of it with \( A_0 \), then

\[
V_{m,n} \cap U_n \neq \emptyset, \quad \forall i \in \mathbb{Z}.
\]

On the other hand, let \( p_k, p_l \) be any fixed points of \( \tilde{\Psi} \), if there exist some up and down transverse horizontal strips, which joint the two points, then there exist infinitely many 1-CO orbits which run from a neighborhood of \( p_k \) to a neighborhood of \( p_l \) along these transverse strips.

Define

\[
\hat{\mathcal{W}} = \{ (m, n) | m \in \mathbb{N}, \ V_m \cap U_n \neq \emptyset \},
\]

\[
\hat{\mathcal{W}} = \{ (m, n) | m \in \mathbb{N}, \ V_n \text{is transverse with } U_m \},
\]

\[
\hat{\Omega} = \{ \omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) | (\omega_0, \omega_{-1}) \in \hat{\mathcal{W}} \}, \quad \forall i \in \mathbb{Z}.
\]
$\Omega = \{\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) | (a_i, a_i+1) \in W, \forall i \in \mathbb{Z}\}$,

$\Omega = \{(N(\Psi(s, \theta)))^i\}_{i=-\infty}^{\infty} | (s, \theta) \in \Lambda\}$.

Theorem 13 is a direct corollary of the following lemma.

Lemma 14.

$\Omega \subseteq \hat{\Omega} \subseteq \Omega$.

Proof. For any $(s, \theta) \in \Lambda$, we have $(s, \theta) \in U_{N(s, \theta)}$ and

$\Psi(s, \theta) \in \Psi(U_{N(s, \theta)}) \cap U_{N(\Psi(s, \theta))}$,

which implies the latter part of this Lemma.

Let $\omega \in \Omega^\infty$, define

$U_{a_{n-k} \cdots a_{n-k}} = U_{a_n} \cap \Psi^{-1}(U_{a_{n-1}}) \cap \cdots \cap \Psi^{-k}(U_{a_k})$,

and set

$U_\omega = \bigcup_{i=0}^{+\infty} \Psi^{-i}(U_{a_i}) = \bigcup_{k=0}^{+\infty} U_{a_{n-k} \cdots a_{n-k}}$,

$V_\omega = \bigcup_{i=1}^{+\infty} \Psi^{-1}(V_{a_{-i}}) = \bigcup_{k=1}^{+\infty} V_{a_{-k} \cdots a_{-k}}$.

Then $U_\omega \subset \hat{\Omega}$ and $V_\omega \subset \hat{\Omega}$ are the up and down horizontal curves, respectively, and intersect transversely. Therefore

$\#(U_\omega \cap V_\omega) = 1$.

Let $(s_\omega, \theta_\omega)$ be the unique element of $U_\omega \cap V_\omega$, then $(s_\omega, \theta_\omega) \in \Lambda$, and

$\{(N(\Psi^{\infty}(s_\omega, \theta_\omega)))^i\}_{i=-\infty}^{\infty} = \omega$.

Thus

$\hat{\Omega} \subseteq \Omega$.

From Theorem 13, we can understand the dynamical behavior of 1-CO orbits of the stadium billiard by geometrical relation of all strips $U_i$ and $V_i$. For example, for any sequence $\omega \in \Omega^\infty$, i.e., $U_\omega$ is transverse with $V_{a_{n+1}}$ for any $i \in \mathbb{Z}$, the stadium billiard possesses a unique 1-CO orbit corresponding to $\omega$, such that the billiard moves in the described manner in Section 1. If $h > h_n$, then for any $m \geq n$, the orbit corresponding to the periodic sequence

$(\ldots, m, m+1, m, m+1, m, \ldots) \in \hat{\Omega}$,

induces a two-periodic point of $\Psi$. Back to the original billiard map $\Psi$, it is a $(2m + 3)$-periodic orbit. Hence, together with Remark 2, for any $K \geq 2n + 2$, there exists a $K$-side inscribed convex polygon, with one vertex at each of the straight segments, and at each of its vertices the tangent will make equal angles with the two sides passing through the vertex. For any $h > 0$, the periodic 1-CO orbits of the stadium billiard are hyperbolic.
3.2. Rotation number of 1-CO orbits

For any \( (s, \theta) \in \Lambda \), the orbit of the billiard started from \( (s, \theta) \) is cyclically ordered. Hence we can define the rotation numbers of these orbits as follows: let \( O \) be the orbit of \( (s, \theta) \) (here \( (s_0, \theta_0) = (s, \theta) \)), the rotation numbers (for two directions) are

\[
\rho_\pm(s, \theta) = \lim_{k \to \pm \infty} \frac{s_k}{|k|L},
\]

(18)

here \( s_k \) is the coordinate in the lift of \( A \) to \( \mathbb{R} \times (-\pi/2, \pi/2) \). Let \( \tilde{O} \) be the intersection of \( O \) with \( A_0 \). Since the limits in (18) always exists for any ordered orbits [10, pp. 424–425], we have

\[
\rho_\pm(s, \theta) = \lim_{m \to \pm \infty} \frac{s_m}{|m|L} = \lim_{m \to \pm \infty} \frac{(|m|/2L)N(\tilde{\Psi}(s, \theta))}{L([m] + \sum_{t=0}^{\pm m} N(\tilde{\Psi}(s, \theta)))}
\]

Set

\[
\tilde{N}_\pm(s, \theta) = \lim_{m \to \pm \infty} \frac{1}{|m|} \sum_{t=0}^{\pm m} N(\tilde{\Psi}(s, \theta)).
\]

Then

\[
\rho_\pm(s, \theta) = (2\tilde{N}_\pm(s, \theta) + 2)^{-1}.
\]

We have, for \( (s, \theta) \in A \), \( \rho_\pm(s, \theta) \leq 1/4 \). Aubry–Mather theory implies that a twist map has quasi-periodic orbits for every frequency in the twist range [1, 8]. The following result shows some further information about the orbits associated with rotation numbers.

Define

\[
A(s, \theta) = \{N(\tilde{\Psi}(s, \theta))\}_{k=-\infty}^{\infty}, \quad \delta_m = [m, m+1), \quad \forall m \in \mathbb{N},
\]

\[
A_\pm = (2(m+1)^{-1}, 2(m+2)^{-1}), \quad \forall m \in \mathbb{N}, \quad A_{\pm, \pm, \pm} = \{(s, \theta) \in A | N(s, \theta) \in \tilde{\Omega}, \rho_\pm(s, \theta) = \rho_\pm\},
\]

\[
A_{\pm, \pm, \pm} = \{(s, \theta) \in A | N(s, \theta) \in \tilde{\Omega}, \rho_\pm(s, \theta) = \rho_\pm, \rho_\pm(s, \theta) \in \Delta_{\pm, \pm, \pm} \}.
\]

Theorem 15. If \( h \geq h_0 \), then for any \( m_+, m_- \geq m \), and \( \rho_\pm \in \Delta_{m_-, \rho_\pm} \), \( \Delta_{m_-, \rho_\pm} \) is dense in \( A_{\pm, \pm, \pm} \).

Proof. Let

\[
\tilde{N}_\pm = \frac{1}{2\rho_\pm} - 1 \in [m, m+1).
\]

Define the averages of a sequence \( \alpha = \{a_k\}_{k=-\infty}^{\infty} \) as

\[
\overline{\alpha} = \lim_{n \to \pm \infty} \frac{1}{|n|} \sum_{k=0}^{\pm n} a_k.
\]

and let

\[
D_{\pm}(\tilde{N}_\pm, \delta_\pm) = N(A_{\pm, \pm, \pm}) = \{\omega \in \tilde{\Omega} | \overline{\alpha} = \tilde{N}_\pm, \overline{\alpha} = \delta_\pm\},
\]

\[
D_{\pm}(\delta_\pm, \delta_\pm) = N(A_{\pm, \pm, \pm}) = \{\omega \in \tilde{\Omega} | \overline{\alpha} \in \delta_\pm, \overline{\alpha} \in \delta_\pm\}.
\]
One can induce a metric of $\Omega^\omega$ by
\[ d^\omega(\omega, \omega') = \sum_{n=0}^{\infty} \frac{|\omega_n - \omega'_n|}{2^n}. \]

Then we only need to prove that $\Omega^\omega_K$ is dense in $\Omega^{[\omega_1, \omega_\omega]}_K$ under the metric. Suppose $\omega \in \Omega^{[\omega_1, \omega_\omega]}_K$, there exist $K_-, K_+ \in \mathbb{N}$ large arbitrarily such that $\omega_{K_-} \leq m_+ \leq \omega_{K_+}$ and $\omega_{K_-} \leq m_- \leq \omega_{K_+}$. Select $a = [a_i]_{i=-\infty}^{\infty}$ be a sequence of 0 and 1, satisfying $a_i = 0$, $-K_- \leq i \leq K_-$, and $a_i = 1(\bar{\omega}_k)$. Define the sequence $a^\omega$ by
\[ a^\omega_k = \begin{cases} 
  m_+ + a_i, & k \leq -K_--1; \\
  a_i, & -K_- \leq k \leq K_+; \\
  m_- + a_i, & k \geq K_+ + 1.
\end{cases} \]

Since $\omega \in \Omega^\omega$, it follows that $(\omega_{K_-}, \omega_{K_+}) \in W^\omega$, i.e., $V_{\omega K}$ is transverse with $V_{\omega K_+}$, which implies that $V_{\omega K}$ is transverse with $V_{m_+}$ by $a_{K_-} \leq m_- \leq a_{K_+}$, and hence $[a^\omega_K, a^\omega_{K+1}) = (a_{K_-}, m_+) \in W^\omega$. From Theorem 1, for any $\delta > 0$, $[a^\omega_K, a^\omega_{K+1}) = (m_- + a_i, m_+ + a_i) \in W^\omega$ for $k \geq K_+ + 1$. Similarly, $(a_{K_-}^\omega, a_{K+}^\omega) = (m_- + a_{K_-}, m_+ + a_{K+}) \in W^\omega$ for $k \leq -K_--1$. Hence, we have $a^\omega \in \Omega^\omega$, and
\[ a^\omega_k = \bar{\omega}_k, \quad \bar{\omega}_k = \bar{\omega}_k. \]

which imply that $a^\omega \in \Omega^{\bar{\omega}}_K$, hence $(a^\omega, \theta_{\omega^*}) = N^{-1}(\omega^*) \in A(\rho_\omega, \rho_\omega)$. Since we can select $K_+, K_-$ large arbitrarily so that $a$ and $a^\omega$ are close to each other arbitrarily, $\Omega^{a^\omega}_K$ is dense in $\Omega^{[\omega_1, \omega_\omega]}_K$. \square

Theorem 15 shows that for any $\rho_\omega \in \Delta_{\omega_{-\omega}}, \rho_\omega \in \Delta_{\omega_{1}}$, there exist infinitely many 1-CO orbits, with the rotation numbers being $\rho_{\omega_1}, \rho_{\omega_\omega}$. Furthermore, such orbits are dense in $A(\rho_{\omega_1}, \rho_{\omega_\omega})$. Hence, one can select two initial values, close to each other arbitrarily, with rotation numbers to be some given numbers. This fact shows that the rotation numbers of 1-CO orbits depend on the initial values sensitively.

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References


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